Differential Geometry III: Riemannian Geometry

9.1 Let $(\mathcal{M}, g)$ be an n-dimensional Riemannian manifold. Recall that, in any local coordinate system, the Ricci tensor satisfies
$$Ric_{ij} = g^{\alpha\beta} R_{\alpha i \beta j}.$$

(a) Show that the Ricci curvature is symmetric, i.e. for any $X, Y \in \Gamma(\mathcal{M})$:
$$Ric(X, Y) = Ric(Y, X).$$

(b) The symmetries of the Riemann curvature tensor imply that not all components R_{ijkl} of the Riemann tensor are independent of each other. How many independent components does R have when $n = 2$? Show that, in this case, for any $X, Y, Z, W \in \Gamma(\mathcal{M})$
$$R(X, Y, Z, W) = K \cdot \big(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\big),$$
where K is the sectional curvature of $\mathcal{M}$ (since $\dim\mathcal{M} = 2$, there is only one tangent 2-plane passing through each point $p \in \mathcal{M}$; hence, in this case, the sectional curvature is simply a function on $\mathcal{M}$).

(c) How many independent components does R have when $n = 3$? Show that, in this case,
$$R_{ijkl} = \mathrm{Ric}_{ik}\, g_{jl} - \mathrm{Ric}_{il}\, g_{jk} + \mathrm{Ric}_{jl}\, g_{ik} - \mathrm{Ric}_{jk}\, g_{il} - \frac{1}{2} S(g_{ik}g_{jl} - g_{jk}g_{il}),$$
where $S = g^{ij}\mathrm{Ric}_{ij}$ is the scalar curvature; in particular, the Ricci curvature contains, in this case, all the information about the Riemann curvature tensor.

9.2 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold and let $\phi : (-\epsilon, \epsilon) \times [0, 1] \to \mathcal{M}$ be a smooth map such that, for each $s \in (-\epsilon, \epsilon)$, $\gamma_s = \phi(s, \cdot)$ is a *geodesic*. Define the vector fields $T = d\phi(\frac{\partial}{\partial t})$ and $X = d\phi(\frac{\partial}{\partial s})$. Prove that
$$\nabla_T \nabla_T X = -R(X, T)T.$$
Intuitively, X measures the infinitesimal separation between nearby geodesics; thus, the Riemann curvature tensor "measures" the relative acceleration of nearby geodesics (compare the behaviour of nearby geodesics in the Euclidean plane vs. the round sphere).

9.3 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold. For any smooth curve $\gamma : [0, 1] \to \mathcal{M}$ and any $t_1, t_2 \in [0, 1]$, we will denote with $\mathbb{P}_{\gamma(t_1)\to\gamma(t_2)} : T_{\gamma(t_1)}\mathcal{M} \to T_{\gamma(t_2)}\mathcal{M}$ the parallel transport along γ from $\gamma(t_1)$ to $\gamma(t_2)$ (with respect to the Levi-Civita connection).

(a) Prove that, for any vector field Z along γ, as $\tau \to 0$:
$$\lim_{\tau\to 0} \frac{Z|_{t=0} - \mathbb{P}_{\gamma(\tau)\to\gamma(0)} Z|_{t=\tau}}{\tau} = -\nabla_{\dot\gamma(0)} Z.$$
Hint: Contruct a frame $\{e_i\}_{i=1}^n$ of vector fields along γ which are parallel translated, and express Z in components with respect to e_i.

*(b) Let $\phi : [-1,1]\times[-1,1] \to \mathcal{M}$ be a smooth map with $p = \phi(0,0)$ and let $X = \phi^*(\frac{\partial}{\partial x^1})$ and $Y = \phi^*(\frac{\partial}{\partial x^2})$. For any $s_1, s_2 \in (0,1)$, we will consider the rectangular loop $\gamma_{(s_1,s_2)}$ starting and ending at p which is of the form $\gamma_{(s_1,s_2)} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned}
\gamma_1(t) &= \phi(t,0), \quad t \in [0,s_1],\\
\gamma_2(s) &= \phi(s_1,s), \quad s \in [0,s_2],\\
\gamma_3(t) &= \phi(s_1 - t, s_2), \quad t \in [0,s_1],\\
\gamma_3(s) &= \phi(0, s_2 - s), \quad s \in [0,s_2].
\end{aligned}$$

For any $Z \in T_p\mathcal{M}$, let $Z_{(s_1,s_2)} \in T_p\mathcal{M}$ be the tangent vector obtained after parallel transporting Z_p around γ, i.e. following the successive mappings

$$\begin{aligned}
Z \to &Z' = \mathbb{P}_{\gamma_1(0)\to\gamma_1(s_1)} Z \to Z'' = \mathbb{P}_{\gamma_2(0)\to\gamma_2(s_2)} Z'\\
&\to Z''' = \mathbb{P}_{\gamma_3(0)\to\gamma_3(s_1)} Z'' \to Z_{(s_1,s_2)} = \mathbb{P}_{\gamma_4(0)\to\gamma_4(s_2)} Z'''.
\end{aligned}$$

Show that

$$\lim_{s_2\to 0}\lim_{s_1\to 0} \frac{Z_{(s_1,s_2)} - Z}{s_1 s_2} = -R(X,Y)Z.$$

Thus, the Riemann curvature tensor quantifies the failure of the parallel transport around small closed loops to be the identity map.